Topology and its Applications 36 (1990) 209-231
North-Holland

C^1 LOCAL PARAMETRIZED MORSE THEORY

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Received 14 April 1988
Revised 24 January 1989

We prove that any k-parameter family of smooth functions on a compact smooth n-manifold can be C^1 approximated by a family of smooth functions having only singularities of "total codimension" $\leqslant \max(1, k-n+1)$.

AMS (MOS) Subj. Class.: Primary 57R45; secondary 57R30

singularities, smooth manifolds, codimension, jets, foliations

Introduction

In this paper we prove the C^1 local version of our results on the elimination of higher singularities in a parametrized family of smooth functions on a compact smooth manifold [3-5]. In the "stable range" where the number of parameters is less than or equal to the dimension of the manifold we get the best possible result since the C^2 local version is false. In the "unstable range" where the dimension of the parameter space is greater than the dimension of the domain manifold, we first prove a new elimination of singularities result and then pass to the C^1 local version.

It was Chaltin who first noticed that a local version of the elimination of higher singularities theorem holds. He showed the following.

Theorem (Chaltin [2]). *Any parametrized family of smooth functions on a compact smooth manifold M which are nonsingular on ∂M can be C^0 approximated by a smooth family of functions having only nondegenerate (A_1) and birth-death (A_2) singularities which are equal to the original functions near ∂M.*

For the sake of completeness we give a short, self-contained proof of a slightly generalized version of this result in Section 3 of this paper.

* This research is supported by NSF Grant No MCS-8502317.

Our results work in the slightly more general C^1 local case. But more importantly, we work in the relative setting. This means we start with a smooth family of smooth functions with no "bad" critical points in a certain region and we perform a C^1 small deformation of this family to eliminate all "bad" singularities without touching the region that we already control. A consequence of this is that our theorems hold in any situation which is locally equivalent to a family of smooth functions on a smooth manifold, for example in the case of a single function on a foliated manifold. In fact this example was what motivated us to consider the C^1 local relative case.

We should point out that the C^r version of Chaltin's theorem is not true for $r \geqslant 2$. For $r \geqslant 3$ this is obvious since A_3 singularities (defined in Section 2 and in [1, 5]) in any generic family of smooth functions are C^3-stable, i.e., they cannot be removed by any C^3 small deformation. A counterexample to the C^2 version of Chaltin's theorem is given by the following 2-parameter family of functions $f: M \to \mathbb{R}$, $t \in P$,

$$f_t(x) = (x_1 + 1)(x_1^2 - 3x_2^2) + t_1 x_1 + t_2 x_2$$

where M and P are both equal to the closed disk of radius 10 in $\mathbb{R}^2$. This family of functions has a circle of degenerate critical points on which there are an odd number of A_3 points. Any C^2 small perturbation of f_t which has only A_1, A_2 and A_3 singularities will necessarily have an odd number of A_3 singularities. Thus they cannot be eliminated.

In Sections 1 and 2 we deal with families of smooth functions in the stable and unstable ranges respectively. The main result of Section 1 is the local framed function theorem (Theorem 1.5). This theorem says that given any smooth family of smooth functions on a compact smooth manifold $f_t: M \to \mathbb{R}$, $t \in D^k$, in the stable range (where $k \leqslant \dim M$), there is an arbitrarily C^1 small deformation of this family which eliminates all singularities except the A_1 and A_2 singularities. Furthermore if there is a compact set of A_1 and A_2 critical points of the original family which admit a structure called a "framing", then the deformation can be taken to be fixed in a neighborhood of this set. One corollary of this theorem is the following.

Corollary 1.10. *Let M be a closed smooth n-manifold and let $\mathcal{F}$ be a smooth foliation of M with codimension $q \leqslant \frac{1}{2}n$. Let $f: M \to \mathbb{R}$ be any smooth function. Then f can be approximated by a smooth function f' which has only A_1 and A_2 singularities when restricted to each leaf of $\mathcal{F}$.*

Section 2 has two theorems. First we show how the proof of the elimination of the higher singularities theorem in [3] can be modified to obtain a good result in the unstable range (Theorems 2.5, 2.6) and then we prove the C^1 local relative version of this new result (Theorem 2.12). The statement of Theorem 2.12 is very technical because the simple concept of a "framing" of an A_1 or A_2 singularity must be replaced by an elaborate construction called a "homotopy framing" for an arbitrary singularity. The consequences of this result can be stated most clearly in the case of a function of a foliated manifold.

Corollary 2.15. *Let M be a closed smooth n-manifold and let $\mathcal{F}$ be a smooth foliation of M with codimension $q > \frac{1}{2}n$. Let $f: M \to \mathbb{R}$ be any smooth function. Then f can be approximated by a smooth function f' having only singularities of total codimension $\leq 2q - n + 1$ on each leaf of $\mathcal{F}$.*

The notion of "total codimension" is developed axiomatically and we show (Proposition 2.3) that the total codimension of any singularity is larger than or equal to its codimension in the sense of Arnol'd [1]. Thus f' in Corollary 2.15 has only singularities of codimension $\leq 2q - n + 1$ in the sense of Arnol'd.

The author is indebted to A. Connes for suggesting to him that a local version of the framed function theorem of [5] should be true and that it could be used to simplify smooth functions on foliated manifolds.

1. The local framed function theorem

The purpose of this section is to prove a local version of the framed function theorem (Theorem 1.4 below) and show how it can be used. The simplest version of the local framed function theorem says that any k-parameter family of smooth functions on a compact smooth manifold M^n can be C^1 approximated by a family of functions having only A_1 and A_2 singularities if $k \leq n$. We recall that A_1 singularities are the nondegenerate Morse singularities and A_2 singularities are the simplest kind of degenerate singularities given by the cubic equation

$$p(x) = x_1^3 \pm x_2^2 \pm x_3^2 \pm \cdots \pm x_n^2. \tag{$*$}$$

(The number of negative signs is called the *index* of the singularity.) We also refer to A_2 singularities as *birth–death* points and we refer to functions with only A_1 and A_2 singularities as *generalized Morse functions.*

Theorem 1.1. *Let M be a compact smooth n-manifold. Let $f_t: M \to \mathbb{R}$, $t \in D^k$, be a smooth family of smooth functions. Then there exists a smooth family of generalized Morse functions $f_t': M \to \mathbb{R}$, $t \in D^k$, so that f_t' is arbitrarily C^1 close to f_t.*

Remarks. (a) When we say that f_t is a smooth family of smooth functions we mean that its adjoint $M \times D^k \to \mathbb{R}$ is smooth (i.e., C^∞).

(b) When we say that f_t' is *arbitrarily C^1 close* to f_t we mean that $\sup|f_t'(x) - f_t(x)| + \sup\|Df_t'(x) - Df_t(x)\|$ is arbitrarily small (with respect to any fixed metric on M), i.e., f_t', f_t are arbitrarily close in $C^0(D^k, C^1(M, \mathbb{R}))$.

(c) Since f_t' is arbitrarily C^1 close to f_t, the singular set of f_t' lies in an arbitrarily small neighborhood of the singular set of f_t. (The *singular set* of f_t is $\Sigma(f_t) = \{(x, t) \in M \times D^k \mid Df_t(x) = 0\}$.) In particular, if f_t is nonsingular on ∂M for all t, then so is f_t'.

(d) To prove this theorem we may assume that f_t is nonsingular on ∂M for all t. Otherwise we can add an exterior collar C to M and extend the family of functions f_t to a new family g_t on $M \cup C$ so that g_t is nonsingular on $\partial(M \cup C)$. We can find a good C^1 approximation g_t' to g_t and take f_t' to be the restriction of g_t' to M.

Theorem 1.1 is not much good the way it is. We need a relative version of Theorem 1.1 which says that if certain singularities of f_t are already of the right kind, then we do not need to change them to construct f_t'. However, this is not a true statement unless these nice singularities of f_t have an additional structure which we call a "framing".

Definition 1.2. A *framing* of a singularity x_0 of a function f (with respect to a given Riemannian metric on M) is defined to be an orthonormal framing ξ of the negative eigenspace of the second derivative of f at x_0.

Definition 1.3. A *framed function* on M is a generalized Morse function f on M together with a framing ξ of each of its critical points. We also require that framed functions be nonsingular on ∂M. By a *family of framed functions* on M parametrized by P we mean a pair (f_t, ξ_t) where $f_t : M \to \mathbb{R}$, $t \in P$, is a smooth parametrized family of generalized Morse functions and ξ_t is a family of functions which for each $t \in P$ and each critical point $x \in M$ of f_t associates an orthonormal framing $\xi_t(x)$ of the negative eigenspace of $D^2 f_t(x)$ (with respect to some fixed metric or smooth family of metrics on M) satisfying the following properties.

(1) f_t is nonsingular on ∂M for all $t \in P$.

(2) f_t is generic in the sense that its 3-jet $j^3 f : M \times P \to J^3(M, \mathbb{R})$ is transverse to the set of degenerate singular 3-jets. In other words, all A_2 points in f_t are universally unfolded. Similarly the restriction of f_t to the boundary of P is assumed to be generic.

(3) If the individual vectors in the framing $\xi_t(x)$ are denoted $\xi_t^1(x), \ldots, \xi_t^i(x)$, then each function ξ^j is smooth on the subset of $M \times P$ on which it is defined.

(4) Suppose that at each A_2 point (x, t) we extend the framing $\xi_t(x) = (\xi_t^1(x), \ldots, \xi_t^i(x))$ to an $(i+1)$-framing by defining $\xi_t^{i+1}(x)$ to be the unique unit vector lying in the one-dimensional null space of $D^2 f_t(x)$ which points in the positive cubic direction (dx_1 in expression $(*)$). If $\bar{\xi}_t$ denotes these extended framings, then the extended functions $\bar{\xi}^j$ are smooth on their domains.

Any single generalized Morse function on M admits a framing if it is nonsingular on ∂M. However a family of generalized Morse functions may not admit a framing (with respect to a family of metrics on M) since the framing is required to vary smoothly (condition (3)). Also when two nondegenerate critical points of index i and $i+1$ are cancelled at a birth-death point of index i then the last framing vector for the index $i+1$ critical point is required to converge to the unit vector in the positive cubic direction of the birth-death point (condition (4)). Lastly a family of framed functions is required to be "generic" (condition (2)).

The main theorem about framed functions is the following.

Theorem 1.4 (framed function theorem [5]). *Let M be a compact smooth n-manifold with a smooth family of metrics parametrized by D^k and let (f_t, ξ_t) be a smooth family of framed functions on M parametrized by S^{k-1} where $k \leq n$. Suppose also that each f_t where $t \in S^{k-1}$ is equal to the same fixed function g near ∂M. Then (f_t, ξ_t) extends to a family of framed functions on M parametrized by D^k so that each f_t for $t \in D^k$ is equal to g near ∂M.*

We can now state the main theorem of this section. This will be the relative version of the theorem stated in the Introduction and the "C^1 local" version of the framed function theorem.

Theorem 1.5 (local framed function theorem). *Let M be a compact smooth n-manifold with a smooth family of metrics parametrized by D^k and let $f_t : M \to \mathbb{R}$, $t \in D^k$, be a smooth family of smooth functions which are nonsingular on ∂M and so that $k \leq n$. Let U be an open subset of $M \times D^k$ so that every singularity of f_t in U is either an A_1 or A_2 singularity and so that the singularities along U admit a framing in a smoothly parametrized way as discussed above. (In particular, f_t is generic along U.) Then there is a family of framed functions (f'_t, ξ_t) on M parametrized over D^k so that*

(a) *f'_t is arbitrarily C^1 close to f_t in the sense of Theorem* 1.1;

(b) *$f'_t - f_t$ has support in an arbitrarily small neighborhood V of $\Sigma(f_t) - U$ in $M \times D^k$ (in particular, $\Sigma(f'_t) - V = \Sigma(f_t) - V$);*

(c) *the framing ξ_t of f'_t along $\Sigma(f'_t) - V$ is equal to the originally given framing of f_t along $\Sigma(f_t) - V$.*

Remarks. (1) When U is empty we get Theorem 1.1. Also Theorem 1.4 follows easily from this theorem.

(2) When U is empty the existence of f'_t satisfying condition (a) implies the existence of f'_t satisfying conditions (a) and (b). This can be seen by "pasting" in the deformation $f_t \to f'_t$ inside V. This means we take a fixed smooth function $\phi : M \times D^k \to [0, 1]$ called a "transition function" so that $\phi = 1$ in a neighborhood of $\Sigma(f_t)$ and $\operatorname{supp}(\phi) \subseteq V$, and we let $f''_t = f_t + \phi(f'_t - f_t)$. If f'_t is sufficiently C^1 close to f_t, then f''_t will be nonsingular in the transition region (the closure of the set on which $0 < \phi < 1$). Thus f''_t will be C^1 close to f_t and $\operatorname{supp}(f''_t - f_t) \subseteq \operatorname{supp}(\phi) \subseteq V$.

Proof. Let $C = \Sigma(f_t) - U$ and let V be an arbitrarily small open neighborhood of C in $M \times D^k$. Then we will deform the family f_t only inside the open set V.

The first deformation will be to make f_t generic. In particular the singularities of the new family f'_t will all be finitely determined. Since f_t is already generic along U, this can be accomplished by an arbitrarily C^∞ small deformation of f_t with support in V.

Let $K = \Sigma(f'_t) \cap \operatorname{supp}(f'_t - f_t) \subseteq V$. Then f'_t is still framed along $\Sigma(f'_t) - K$. For each $(x_0, t_0) \in K$ we choose a small elevator neighborhood E_0 of x_0 in M and we extend this to a parametrized elevator (E_t, a, b, ϕ_t) which is contained in V. [3, Section 4] explains the definition and construction of these elevators, however the precise definition of elevators is not important. We only need the following properties.

(E0) E_0 is a compact, codimension-zero, smooth submanifold of $\operatorname{int} M$ (with corners).

(E1) f_{t_0} has no singularities on ∂E_0.

(E2) ϕ_t is a smooth family of embeddings $E_0 \to M$ parametrized by a compact k-manifold neighborhood B of t_0 in D^k and $\phi_t(E_0) = E_t$ for all $t \in B$.

(E3) The functions $f_t\phi_t$ and $f_{t_0}|E_0 : E_0 \to \mathbb{R}$ agree in a neighborhood of ∂E_0.

(E4) $f_t(E_t) \subseteq [a, b]$ for all $t \in B$. ($b - a$ is called the *height* of the elevator.)

We use the notation $E = \{(x, t) \mid t \in B, x \in E_t\}$ and we call E a parametrized elevator neighborhood of (x_0, t_0).

For each $t_0 \in D^k$ we consider the critical points $x_1, x_2, \ldots, x_m$ of f_{t_0} which lie in the set K. We take the parametrized elevator neighborhoods $E^1, \ldots, E^m$ of $(x_1, t_0), \ldots, (x_m, t_0)$ in V as constructed above. By making the parameter domains $B^1, \ldots, B^m$ smaller we may assume that $B^1 = B^2 = \cdots = B^m = B_{t_0}$ and that all the points in K which lie over B_{t_0} are contained in some E^i. We call the elevators $E^1, \ldots, E^m$ a *stack of elevators* over B_{t_0}. Since D^k is compact, there is a finite set of points $t_1, \ldots, t_r$ so that the interiors of the sets $B_{t_1}, \ldots, B_{t_r}$ cover D^k.

Now choose a smooth triangulation of D^k so that each closed simplex lies in the interior of some B_{t_i}. The desired family of functions f_t' is constructed by deforming f_t over the simplices of D^k starting with the simplices of lowest dimension.

If $v \in D^k$ is a vertex, then $v \in \operatorname{int} B_{t_i}$ for some i. Let $E^1, \ldots, E^m$ be the elevators over B_{t_i}. Let $\Sigma_1 = \Sigma(f_v) \cap K$ and let $\Sigma_0 = \Sigma(f_v) - K$. Then, by construction the critical points of f_v in Σ_0 are already equipped with a framed structure. By an arbitrarily C^∞ small deformation of the family f_t with support in an arbitrarily small neighborhood W of Σ_1 in $E^1 \cup \cdots \cup E^m$ we get a new family which we also call f_t so that f_t has only nondegenerate singularities in W for t near v. We then choose an arbitrary smooth family of framings for these singularities. This gives f_t the structure of a framed function for t in a small neighborhood of each vertex.

Now let Δ^p be a p-simplex of D^k. Then $\Delta^p \subseteq \operatorname{int} B_{t_i}$ for some i. Let $E^1, \ldots, E^m$ be the elevators over B_{t_i}. Let Σ be the part of the singular set of f_t which lies over Δ^p and inside $E = E^1 \cup \cdots \cup E^m$. Let Σ_0 be the union of components of Σ consisting of generic families of A_1 and A_2 singularities on which framings have already been defined. Let $\Sigma_1 = \Sigma - \Sigma_0$. Note that by induction on p, the functions f_t are framed for t in a neighborhood of $\partial\Delta^p$.

If $p < k$, then by induction on k our theorem is true over Δ^p. Consequently there is an arbitrarily C^1 small deformation f_t^u of $f_t|\Delta^p$ with support in an arbitrarily small neighborhood of Σ_1 so that $f_t|\Delta^p$ becomes framed. By radiating this deformation [3, Section 4] or by "pasting in" an extension of this deformation we can make f_t framed over a small neighborhood of Δ^p.

This "pasting" construction works as follows. Let $\bar{f}_t^u$ be an extension of the deformation f_t^u to all of D^k. Then since f_t^u is a C^1 small deformation over Δ^p, $\bar{f}_t^u$

will be a C^1 small deformation in a small neighborhood V of Δ^p. Also we may assume that $\bar{f}_t^u$ is the trivial deformation over a neighborhood of $\partial\Delta^p$. Let $\psi: D^k \to [0,1]$ be any smooth function with support in V so that $\psi = 1$ on Δ^p. Then the deformation $\psi\bar{f}_t^u + (1-\psi)f_t$ is what we want. It is obtained from f_t by pasting in an extension of f_t^u.

If $p = k$, then we ignore the distinction between Σ_0 and Σ_1. We take the portion of each elevator E^i which lies over $\Delta^p = \Delta^k$ as input for the framed function theorem. We then get new families of generalized Morse functions $g_t^i: E_t^i \to \mathbb{R}$, $t \in \Delta^k$, so that $g_t^i = f_t$ near $\partial(E^i | \Delta^k)$ which admit framings which extend the already given framings for t near $\partial\Delta^k$. Now let $f_t' = f_t$ outside the union $\bigcup (E^i | \Delta^k)$ and let $f_t' = g_t^i$ on $E^i | \Delta^k$. The new family f_t' is framed over Δ^k.

There is one last point to take care of. We must choose the functions g_t^i so that f_t' is arbitrarily C^1 close to f_t. To do this we first choose the heights $(b-a)$ of our elevators to be sufficiently small so that f_t' is C^0 close to f_t. Then we only need to control the derivative of $f_t' - f_t$. Since the elevators lie inside a small neighborhood of the singular set of f_t we can assume that the derivative of f_t is very small inside the elevators. This means we only need to show that the derivatives of g_t^i are very small, or equivalently bounded, as functions of $b-a$. This can be accomplished by choosing one g_t^i and "squeezing" it.

First we have to choose our original elevators (E, a, b) so that there is a smaller elevator (F, c, d) so that $c = a + \frac{1}{3}(b-a)$, $d = a + \frac{2}{3}(b-a)$ and F_t contains all the singularities of f_t in E_t. Then we can do all of our deformations inside the smaller elevators F_t until we come to the last step. After the construction of g_t^i we can compose $g_t^i | F_t$ with an affine linear function $T_t: \mathbb{R} \to \mathbb{R}$ with a very small slope so that the derivative of $Tg_t^i | F_t$ is very small. Using a transition function we can extend $Tg_t^i | F_t$ to all of E_t so that it agrees with f_t near ∂E_t. This gives a new g_t^i whose derivative is arbitrarily small in F_t and approximately $\frac{3}{2}$ of the derivative of f_t on $E_t - F_t$. The framing of g_t^i will give a framing for the new g_t^i. This completes the proof of the local framed function theorem. □

We now give three examples of how the local framed function theorem can be used.

Corollary 1.6. *Let $M^n \to E^{n+k} \to B^k$ be a smooth bundle of paracompact smooth manifolds of the indicated dimensions where $k \le n$. Let $f: E \to \mathbb{R}$ be a smooth function so that on each fiber M, $f | M$ is nonsingular on ∂M. Then f can be strongly C^0 approximated by a smooth function $f': E \to \mathbb{R}$ so that f' is a generalized Morse function on each fiber. In fact f' admits the structure of a family of framed functions on the fibers. Furthermore f' can be chosen so that the derivative of $f' - f$ in the fiber direction is less than any positive function $\varepsilon: E \to (0, \infty)$.*

Proof. Choose a locally finite covering of E by compact product manifolds $A_\alpha \times C_\alpha$ where each C_α is diffeomorphic to D^k. Choose a well ordering of the index set $\{\alpha\}$ and, starting with the smallest α, deform f_t with support in the interior of $A_\alpha \times C_\alpha$

so that it becomes framable on most of the interior of $A_\alpha \times C_\alpha$. By transfinite induction and Theorem 1.5 we can deform f_t so that it becomes framable on a larger and larger set. (Remark (d) after Theorem 1.1 tells how to get around the fact that f_t may be singular on $\partial A_\alpha \times C_\alpha$.) Since the covering is locally finite we eventually get a family of generalized Morse functions which is everywhere framable. □

Corollary 1.7. *Let $M^n \to E^{n+k} \to B^k$ be a smooth bundle of paracompact smooth manifolds of the indicated dimensions where $k \leq n$. Then there is a smooth function $f: E \to \mathbb{R}$ so that on each fiber M, $f|M$ is a generalized Morse function which is nonsingular on ∂M.*

Proof. For each $e \in E$ let $g(x)$ be the distance from x to $\partial p^{-1}p(x)$. Then g can be smoothed out to give a smooth function on E which is nonsingular on the boundary of each fiber. Corollary 1.6 now says that g can be approximated by a function f as desired. □

Let $f: M \to S^1$ be a smooth function. Then f looks locally like a smooth function from M to $\mathbb{R}$. In fact there is a covering map $\tilde{f}: \tilde{M} \to \mathbb{R}$ over f where $\tilde{M}$ is a covering manifold of M. Consequently it makes sense to talk about a "framing" for the function f. More precisely, suppose that f is nonsingular on ∂M and f has only A_1, A_2 singularities. Then we define a *framed circle function* structure on f to be a covering translation invariant framed function structure on $\tilde{f}$. We will assume that M has a fixed metric and that $\tilde{M}$ has the induced metric. Note that the metric on $\tilde{M}$ depends only on the homotopy class of f.

Corollary 1.8. *Let M^n be a compact smooth n-manifold with a fixed metric. Let $f_t: M \to S^1$, $t \in D^k$, be a smooth family of smooth functions which are nonsingular on ∂M and suppose $k \leq n$. Suppose that f_t has the structure of a smooth family of framed circle functions for t near ∂D^k. Then there is a smooth family of functions $f'_t: M \to S^1$ so that*

(a) *f'_t is arbitrarily C^1 close to f_t,*

(b) *$f'_t = f_t$ near ∂D^k,*

(c) *f'_t has only A_1, A_2 singularities and*

(d) *f'_t has the structure of a smooth family of framed circle functions extending the already given framed structure over $\partial \Delta^k$.*

Proof. For each $t_0 \in D^k$ we choose a regular value c of f_{t_0}. Then c is also a regular value of f_t for each t in a compact neighborhood B of t_0. Choose a finite set $B_1, B_2, \ldots, B_m$ of such B so that the interiors of the B_i cover D^k. Choose a smooth triangulation of D^k so that every closed simplex lies in the interior of some B_i.

By induction on p we may assume that f_t has the structure of a framed circle function for t in a neighborhood of the $(p-1)$-skeleton of D^k. Let Δ^p be a p-simplex.

Then $\Delta^p \subseteq \operatorname{int} B_i$ for some i. By cutting the domain M open along the associated regular value c_i we get a family of compact manifolds $M_t = f_t^{-1}(S^1 - c_i)$ union two copies of $f_t^{-1}(c_i)$. $f_t \mid \Delta^p$ can be viewed as a family of maps $g_t : M_t \to [c_i, c_i + 1]$. Since Δ^p is contractible, the M_t are diffeomorphic to a fixed manifold and we can use the local framed function theorem to approximate g_t by a family of framed functions g_t'. By pasting in an extension of g_t' to a neighborhood of Δ^p in B_i we can change f_t so that it is framed over Δ^p. By Theorem 1.5(b) we can make sure that we only change f_t in a very small neighborhood of $\Sigma(f_t)$ which avoids all the slice sets $\{(x, t) \in M \times D^k \mid t \in B_i \text{ and } f_t(x) = c_i\}$. □

Corollary 1.9. *Let M be a compact smooth n-manifold without boundary. Let $\mathcal{L}^\Delta(M, S^1)$ be the simplicial set of Δ^k families of framed circle functions on M. Then the natural map $|\mathcal{L}^\Delta(M, S^1)| \to C^0(M, S^1) = (S^1)^M$ is n-connected.*

Remark. If M has a boundary, then we can fix all maps on ∂M and we get $|\mathcal{L}^\Delta(M, S^1)| \simeq_n C^0(M, S^1 \text{ rel } \partial M)$.

Proof. Left to the reader. (First show that $\mathcal{L}^\Delta(M, S^1)$ is a Kan complex as [5, Proposition 1.4].) □

Corollary 1.10. *Let M be a compact smooth n-manifold and let $\mathcal{F}$ be a smooth foliation of M with codimension $q \leq \frac{1}{2}n$. Let $f : M \to \mathbb{R}$ be a smooth function satisfying the following boundary condition:*

> *If $x \in \partial M$ and the derivative of f at x is trivial on the tangent plane to the leaf at x, then $\mathcal{F}$ is tangent to ∂M in a neighborhood of x.*

Then f can be approximated by a smooth function f' which satisfies the same boundary condition so that f' is a generalized Morse function on each leaf of $\mathcal{F}$. Furthermore, f' admits the structure of a family of framed functions on the leaves of $\mathcal{F}$. And f' can be chosen so that the first derivative of $f' - f$ is uniformly small in the direction tangent to the leaves of $\mathcal{F}$.

Remark. (a) The analogous result holds for maps $M \to S^1$.

(b) If $\mathcal{F}$ is tangent to ∂M, then any function $f : M \to \mathbb{R}$ satisfies the boundary condition so f' always exists.

Proof. (This is very similar to the proof of the local framed function theorem itself.) First note that the boundary condition is an open condition on f and its first derivative in the direction of the leaves. Thus any C^1 approximation of f will satisfy the boundary condition.

Next we can do an arbitrarily C^∞ small deformation of f to make it generic. In particular the singularities of f on the leaves of $\mathcal{F}$ will be finitely determined. If $\Sigma(f)$ is the set of singularities of f considered as a function on the leaves, then

$\Sigma(f)$ will be a compact smooth q-manifold where q is the codimension of $\mathcal{F}$ and $\partial\Sigma(f)=\Sigma(f)\cap\partial M$.

For each $x_0\in\Sigma(f)$ we can find a product neighborhood N of x_0 in M and a diffeomorphism $\phi: D^{n-q}\times I^q\to N$ so that $x_0=\phi(0, t_0)$ for some $t_0\in I^q$ and $\phi(D^{n-q}\times t)$ is tangent to $\mathcal{F}$ (i.e., lies in one leaf) for all $t\in I^q$. By choosing N suitably small (choose D^{n-q} first and then I^q) we may assume that the restriction of $f\phi$ to $D^{n-q}\times t$ is nonsingular on $\partial D^{n-q}\times t$ for all t in I^q.

Since $\Sigma(f)$ is compact we can find a finite set of such product neighborhoods $N_1,\ldots, N_m$ so that their interiors cover $\Sigma(f)$. Using the local framed function theorem on the restriction of f to each N_i we can gradually make f more and more into a framed function. Thus the first step is to change f inside N_1 so that we get a new function f_1 which has an open set of singularities U_1 which are generic A_1 and A_2 singularities together with a smooth family of framings and so that $\Sigma(f_1)-U_1$ lies in the union of the interiors of $N_2,\ldots, N_m$. Next we change f_1 inside N_2 and we get a new function f_2 with an open set of framed singularities U_2 so that $\Sigma(f_2)-U_2$ lies in the union of the interiors of $N_3,\ldots, N_m$. Although U_2 may not contain U_1, by the local framed function theorem we can make U_2 contain a neighborhood W in U_1 of an arbitrarily large compact subset of U_1 and the framing of f_2 on W will be the same as the framing of f_1. After m steps the function f will be completely framed. $\square$

2. Families of functions in the unstable range

The unstable range is where the number of parameters is greater than the dimension of the manifold. In our paper [3] we did not maintain good control over the singularities beyond the stable range. However by modifying the proof in [3] very slightly we can get a useful statement about the singularities that one gets in the unstable range. The simplest version of the theorem in the unstable range says that any k-parameter family of smooth functions on a compact smooth n-manifold M^n where $k>n$ can be C^1 approximated by a family of functions having only singularities of total codimension $\leqslant k-n+1$. The notion of "total codimension" is defined axiomatically below.

Theorem 2.1. *Let M be a compact smooth n-manifold and let $f_t: M\to\mathbb{R}$, $t\in D^k$, be a smooth family of functions where $k>n$. Then there exists another smooth family of functions f_t' on M which is arbitrarily C^1 close to f_t so that f_t' has only singularities of total codimension $\leqslant k-n+1$.*

Remarks. (1) This theorem holds for any choice of the notion of total codimension which satisfies the axioms (TC1)-(TC6) listed below.

(2) By Proposition 2.3 below the total codimension of any singularity is larger than or equal to its codimension in the sense of Arnol'd [1]. Thus there is a C^1

approximation f_t' for f_t which has only singularities of codimension $\leqslant k-n+1$ in the sense of Arnol'd.

(3) The weaker version of this theorem with C^1 replaced by C^0 follows from a theorem of Chaltin (see Section 3).

We will not give the proof of this theorem since it follows from the relative version (Theorem 2.12) proved below.

As an example of the theorem suppose that $n=10$ and $k=12$. Then $k-n+1=3$ so f_t' will have a 12-dimensional singular set with an 11-dimensional birth-death set, a 10-dimensional dovetail (A_3) set and a 9-dimensional set of singularities of codimension 3 consisting of butterflies (A_4) and hyperbolic and elliptic umbilics (both called D_4 since they are equivalent over $\mathbb{C}$.) In general A_k *singularities* are given by

$$\pm x_1^{k+1} + \sum_{i=2}^{n} \pm x_i^2$$

and D_k *singularities* are given by

$$x_1^2 x_2 \pm x_2^{k-1} + \sum_{i=3}^{n} \pm x_i^2.$$

D_4 is called the *hyperbolic* or *elliptic umbilic* depending on whether the sign in front of $x_2^{k-1} = x_2^3$ is $+$ or $-$ respectively. (See [1] for a list of all singularities of codimension $\leqslant 10$ in the sense of Arnol'd.)

We now give our axiomatic definition of "total codimension". Let $\mathbb{C}J^s(n) = \mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{s+1}$ with unique maximal ideal $\mathbb{C}\mathcal{M}_s$ if $s<\infty$ and let $\mathbb{C}J^\infty(n) = \mathbb{C}[[x_1, x_2, \ldots, x_n]]$ with unique maximal ideal $\mathbb{C}\mathcal{M}$. Then we define total codimension to be any function $\operatorname{Tcod}: \mathbb{C}\mathcal{M}^2 \to \mathbb{N} \cup \{\infty\}$ satisfying the six axioms (TC1)-(TC6) listed below. The *total codimension* of a singularity x_0 of a function f will be defined to be the total codimension of the Taylor series of the function $f-f(x_0)$ at x_0 considered as a power series over $\mathbb{C}$.

(TC1) (finite determination axiom) $\operatorname{Tcod}(p)$ *is finite if and only if p is finitely determined, or equivalently,* 0 *is an isolated singularity of any smooth function with Taylor series equal to p at* 0.

(TC2) (upper semicontinuity axiom) *For each $i<\infty$, $T_i = \{p \in \mathbb{C}\mathcal{M}^2 \mid \operatorname{Tcod}(p) \geqslant i\}$ is the complete inverse image under the natural map $\mathbb{C}\mathcal{M}^2 \to \mathbb{C}\mathcal{M}_s^2$ of an algebraic subset C_i of $\mathbb{C}\mathcal{M}_s^2 \subseteq \mathbb{C}J^s(n)$ for some $s<\infty$. (We use the notation $s=s(i-1)$.)*

(TC3) (codimension axiom)

(a) *Each irreducible component of each T_i has codimension $\geqslant i$ in $\mathbb{C}\mathcal{M}^2$. (In other words, each irreducible component of each C_i has codimension $\geqslant i$ in $\mathbb{C}\mathcal{M}_s^2$.)*

(b) *Each irreducible component of T_i which does not lie in T_{i+1} has codimension exactly equal to i in $\mathbb{C}\mathcal{M}^2$ for each i.*

(TC4) (locally constant multiplicity axiom) *The multiplicity function $\mu(p)$ = degree at zero of $\nabla p: \mathbb{C}^n \to \mathbb{C}^n$ is locally constant on $T_i - T_{i-1}$ in the inverse limit Zariski topology for all finite i.*

(TC5) (right invariance) *Each T_i is invariant under the right action of* $\mathrm{Aut}_{\mathbb{C}}\, J^\infty(n)$.

(TC6) (left scalar invariance) *Each T_i is also invariant under left multiplication by nonzero complex numbers.*

The multiplicity of an isolated singularity is also known as its "Milnor number". It is equal to the number of nondegenerate singularities that result from a generic perturbation of the singularity (see [7, Appendix B]). It is well known that the multiplicity of an isolated singularity is always exactly one more than its orbital codimension where the *orbital codimension* of $p \in \mathbb{C}\mathcal{M}^2$ is defined to be the codimension of $\mathbb{C}\mathcal{M}^2$ of the orbit of p under the action of the automorphism group of $\mathbb{C}J^\infty(n)$. The orbital codimension of p is given by the formula

$$\mathrm{Ocod}(p) = \dim_{\mathbb{C}} \mathbb{C}\mathcal{M}/J(p) = \dim_{\mathbb{C}} \mathbb{C}\mathcal{M}^2/\mathbb{C}\mathcal{M}J(p)$$

where $J(p)$ is the *Jacobian ideal of p* which is the ideal in $\mathbb{C}J^\infty(n)$ generated by the partial derivatives $(\partial/\partial x_i)p$. This well-known fact follows easily from the observation that the tangent plane at p to the orbit of p is $\mathbb{C}\mathcal{M}J(p)$. It follows from this formula that orbital codimension is upper semicontinuous on $\mathbb{C}\mathcal{M}^2$, in fact the set of all $p \in \mathbb{C}\mathcal{M}^2$ so that $\mathrm{Ocod}(p) \geq c$ is equal to the inverse image of an algebraic subset of $\mathbb{C}J^{c+1}(n)$ (see, e.g., [3, A2.5]). In the case of nonisolated singularities all of these numbers are defined to be $+\infty$.

Suppose that $p \in \mathbb{C}\mathcal{M}^2$ is an isolated singularity with multiplicity μ. Then the *modality m* of p is defined to be the codimension of the orbit of p in the space of all isolated singularities with the same multiplicity. Arnol'd's notion of the codimension of p is given by $\mathrm{Acod}(p) = \mu - m - 1$. Since the nonsingular points in a complex variety are dense, the algebraic and topological notions of codimension agree. Thus $\mathrm{Acod}(p)$ is the smallest codimension in $\mathbb{C}\mathcal{M}^2$ of an irreducible component of the space of all singularities of multiplicity $\geq \mu$ which contains p. Note that $\mathrm{Acod}(p) \leq \mathrm{Ocod}(p)$ for all p.

Proposition 2.2. *For any $n \geq 0$ there exists a total codimension function* $\mathrm{Tcod}: \mathbb{C}\mathcal{M}^2 \to \mathbb{N} \cup \{\infty\}$.

Proof. Let (TC1a) denote the following weaker form of (TC1).

(TC1a) *Each element of $T_i - T_{i+1}$ is finitely determined if $i < \infty$.*

Let $T_0 = \mathbb{C}\mathcal{M}^2$ and let $T_1 \subseteq \mathbb{C}\mathcal{M}^2$ be defined to be the set of all degenerate singularities. Then (TC2), (TC3)(a), (TC5), (TC6) hold for $i = 0, 1$ and the axioms (TC1a), (TC3)(b), (TC4) hold for $i = 0$. Suppose by induction that $T_0, T_1, \ldots, T_c$ are defined so that (TC2), (TC3)(a), (TC5), (TC6) hold for $i \leq c$ and the axioms (TC1a), (TC3)(b), (TC4) hold for $i < c$. Then we will construct T_{c+1} so that these axioms hold for the next i. We then claim that the remaining axiom, (TC1), follows from these axioms. To see this suppose that $p \in \mathbb{C}\mathcal{M}^2$ with $\mathrm{Tcod}(p) < \infty$. Then by (TC5) and (TC3)(a), the orbital codimension of p is infinite so p is not finitely determined. The converse follows from (TC1a).

We now construct T_{c+1}. Let $C_c \subseteq \mathbb{C}\mathcal{M}_s^2$ be as given by (TC2) and let $A_1, \ldots, A_m$ be the irreducible components of C_c. Let Σ be the union of

(1) all A_j with codimension $>i$ in $\mathbb{C}\mathcal{M}_s^2$,

(2) all singular points of all A_j and

(3) all intersections $A_j \cap A_k$.

Then $C_c - \Sigma = \coprod B_j$ where $B_j = (C_c - \Sigma) \cap A_j$. Note that each B_j is a nonsingular quasiaffine variety of codimension i in $\mathbb{C}\mathcal{M}_s^2$. We will consider only those subscripts j for which B_j is nonempty.

Let s_j be the minimal orbital codimension of an element of the inverse image $\pi^{-1}B_j$ of B_j in $\mathbb{C}\mathcal{M}^2$. Let U_j be the set of all $p \in \pi^{-1}B_j$ so that $\mathrm{Ocod}(p) = s_j$. Then s_j is finite and U_j is the complete inverse image of an open subset of $\mathbb{C}\mathcal{M}_t^2$ where $t = \max(s, s_j + 2)$. This is because $\mathrm{Ocod}(p) \leqslant s_j$ if and only if the orbital codimension of the $(s_j + 2)$-jet of p is $\leqslant s_j$. We define T_{c+1} to be $T_c - (\coprod U_j)$.

The axioms are satisfied by construction of T_{c+1}. (TC1) for $i = c$ follows from the fact that each s_j must be finite. (TC2) holds for $i = c+1$ since T_c is the inverse image of a closed subset of $\mathbb{C}\mathcal{M}_t^2$. (TC3)(a) for $i = c+1$ and (TC3)(b) for $i = c$ hold since we removed a nonempty Zariski open subset of each component of T_c of codimension i. (TC5) and (TC6) hold for $i = c+1$ since T_{c+1} was intrinsically defined. And (TC4) holds for $i = c$ by construction. □

Proposition 2.3. *Suppose that* $\mathrm{Tcod}: \mathbb{C}\mathcal{M}^2 \to \mathbb{N} \cup \{\infty\}$ *is any total codimension function. Then for any* $p \in \mathbb{C}\mathcal{M}^2$ *we have*

$$\mathrm{Ocod}(p) \geqslant \mathrm{Tcod}(p) \geqslant \mathrm{Acod}(p).$$

Corollary 2.4. *Suppose that* $\mathrm{Tcod}: \mathbb{C}\mathcal{M}^2 \to \mathbb{N} \cup \{\infty\}$ *is any total codimension function. Then for any* $c \leqslant 5$, *the three sets* $\mathrm{Ocod}^{-1}(c)$, $\mathrm{Tcod}^{-1}(c)$, $\mathrm{Acod}^{-1}(p)$ *are equal.*

Proof. It is well known that $\mathrm{Ocod}^{-1}(c) = \mathrm{Acod}^{-1}(c)$ if $c \leqslant 5$. But Proposition 2.3 says that $\mathrm{Ocod}^{-1}\{0, \ldots, c\} \subseteq \mathrm{Tcod}^{-1}\{0, \ldots, c\} \subseteq \mathrm{Acod}^{-1}\{0, \ldots, c\}$. □

Proof of Proposition 2.3. Suppose that $\mathrm{Tcod}(p) = c$. Then by the right invariance axiom, $T_c - T_{c+1}$ contains the orbit of p. So $\mathrm{Ocod}(p)$ is greater than or equal to the codimension of $T_c - T_{c+1}$ in $\mathbb{C}\mathcal{M}^2$. But this is equal to c by the codimension axiom.

The locally constant multiplicity axiom implies that the modality m of p is at least equal to the codimension of the orbit of p in $T_c - T_{c+1}$. In other words $m \geqslant \mathrm{Ocod}(p) - \mathrm{Tcod}(p)$ which is equivalent to $\mathrm{Tcod}(p) \geqslant \mathrm{Ocod}(p) - m = \mathrm{Acod}(p)$. □

Now we will state and prove the new version of the elimination of the higher singularities theorem. This theorem states that a k-parameter family of functions on a smooth n-manifold where $k > n$ can be deformed so that the singularities of total codimension $\geqslant k - n + 2$ can be eliminated. Unfortunately we need a technically much more complicated version of this theorem in order to derive the C^1 local version from it. We need some notation.

Let n and c be fixed positive integers (e.g., $n = \dim M$ and $c = k - n + 1$) so that we are trying to eliminate singularities of total codimension $>c$ from families of functions on n-manifolds. Let s be an integer greater than or equal to the integer $s(c)$ given by axiom (TC2) and let $J^s(n)$ be the $\mathbb{R}$-algebra of all real polynomials of degree $\leq s$ in n variables. Let $J^s_c(n)$ be the set of all $p \in J^s(n)$ so that p is either nonsingular at 0 or 0 is a singularity of p of total codimension $\leq c$. Whether a singularity has total codimension $\leq c$ is determined by its $s(c)$-jet and thus by its s-jet by (TC2) so this makes sense. Let $J^s_*(n)$ be the set of all $p \in J^s(n)$ so that either p is nonsingular at 0 or 0 is a nondegenerate singularity of p of index 0 (in particular a relative minimum). Then $J^s_*(n)$ is a contractible subset of $J^s_c(n)$. (The space $J^s_*(n)$ linearly contracts to the single point $x_1^2 + \cdots + x_n^2$.)

If M is a smooth n-manifold, then let $J^s(M)$ be the space of all s-jets of maps $M \to \mathbb{R}$. This is a smooth bundle over M with fiber $J^s(n)$ and group $\operatorname{Aut}_{\mathbb{R}} J^s(n)$. Since $J^s_c(n)$ and $J^s_*(n)$ are invariant subspaces of $J^s(n)$ they define subbundles $J^s_c(M)$ and $J^s_*(M)$ of $J^s(M)$. Any smooth map $f: M \to \mathbb{R}$ gives a section of the bundle $J^s(M)$ which is called the *s-jet* of f and which is denoted $j^s f$. If the singularities of f are all of total codimension $\leq c$, then $j^s f$ is a section of $J^s_c(M)$.

Now let M be a fixed compact smooth manifold and let $g: M \to \mathbb{R}$ be a fixed smooth function which is nonsingular on ∂M. For every positive integer c let $\mathcal{H}_c(M)$ be the space of all smooth functions on M which are equal to g near ∂M and all of whose singularities have total codimension $\leq c$. Let $\Gamma_c(M)$ be the space of all sections σ of the jet bundle $J^{s(c)}_c(M)$ over M so that $\sigma = j^{s(c)} g$ over ∂M. Then the new version of the higher singularities theorem is as follows.

Theorem 2.5 (higher singularities theorem). *The $s(c)$-jet map*

$$j^{s(c)}: \mathcal{H}_c(M) \to \Gamma_c(M)$$

is $(n+c-1)$-connected if $c \geq 1$.

It is easy to see that this theorem is equivalent to the following theorem when $L = \partial D^k$.

Theorem 2.6 (higher singularities theorem: technical version). *Let $f_t: M \to \mathbb{R}$, $t \in D^k$, be a smooth family of smooth functions so that $f_t = g$ near ∂M for all t. Let L be a compact subset of D^k and suppose that f_t has only singularities of total codimension $\leq c$ for all $t \in L$ where c is a fixed positive integer $\geq k - n + 1$. Let g^u_t be a deformation of $g^0_t = j^{s(c)} f_t$ into $J^{s(c)}_c(M)$ which keeps L in $J^{s(c)}_c(M)$, in other words*

(a) *g^u_t is a section of $J^{s(c)}_c(M)$ for all $(t, u) \in L \times I$,*

(b) *g^1_t is a section of $J^{s(c)}_*(M)$ for all $t \in L$ and*

(c) *$\operatorname{supp} g^u_t \subseteq \operatorname{int} M$ for all $t \in D^k$.*

Then there exists a deformation h^u_t of f_t with support in $(\operatorname{int} M) \times (D^k - L)$ and a deformation $\bar{g}^u_t$ of $\bar{g}^0_t = h^1_t$ so that

(1) *$\bar{g}^u_t(M) \subseteq J^{s(c)}_c(M)$ for all $(t, u) \in D^k \times I$,*

(2) *$\bar{g}^1_t(M) \subseteq J^{s(c)}_*(M)$ for all $t \in D^k$ and*

(3) $\operatorname{supp} \bar{g}_t^u \subseteq \operatorname{int} M$ *for all* $t \in D^k$.

Note that (1) *implies that* h_t^1 *has only singularities of total codimension* $\leq c$ *for all* $t \in D^k$.

Remark. It suffices to prove this in the case $L = \partial D^k$. In the general case we know that f_t has singularities of total codimension $\leq c$ for t in a neighborhood of L and g_t^u satisfies (a) and (b) for t in a neighborhood of L since $J_c^{s(c)}(M)$ and $J_*^{s(c)}(M)$ are open in $J^{s(c)}(M)$. Using the special case of the theorem for $k = j$ we can construct h_t^u, g_t^u over $L \cup H$ where H is a j-handle. After a finite number of steps we get h_t^u, $\bar{g}_t^u$ defined over all of D^k.

Proof. In [3] we used a noninvariant refinement of the Thom–Boardman stratification of the jet space $J^\infty(n)$. However we can change the construction of this noninvariant stratification so that it is a refinement of a stratification based on total codimension.

The noninvariant stratification is given in [3, Definition 1.6]. We recall that we have a translate $S^I = \Sigma^I + P$ of the Thom–Boardman stratum Σ^I and X is a closed semialgebraic subset of $[0, \infty) \times S^I$ which contains $0 \times S^I$. Let $\pi : X \to S^I$ be the projection map and let $d = \dim S^I$. The main property of X is that for each $x \in S^I$, $(0, x)$ is isolated in $\pi^{-1}(x)$. Now let $V_i \subseteq S^I$ be defined in a new way as follows. (We identify S^I with $0 \times S^I$ to simplify notation.) If $i = 0$, then let $V_0 = S^I$. Now suppose by induction that V_i is defined so that V_i is a closed semialgebraic set of dimension $\leq d - i$. Let $W_i = \pi^{-1}(V_i)$ and note that $\dim W_i \leq d - i + 1$. Then V_{i+1} is defined by $V_{i+1} = (V_i \cap W_i^{d-i-1}) \cup \Sigma_{\mathrm{TC}} V_i$ where W_i^{d-i-1} is as in the notation of [3] and $\Sigma_{\mathrm{TC}} V_i$ is defined as follows. Let $A_1, \ldots, A_m$ be the irreducible components of V_i and let Σ be the union of

(1) all A_j of dimension $< d - i$,

(2) all singular points of all A_j and

(3) all intersections $A_j \cap A_k$ where $j \neq k$.

Then $V_i - \Sigma = \coprod B_j$ where $B_j = (V_i - \Sigma) \cap A_j$. For each j let U_j be the set of all $x \in B_j$ with minimal total codimension in B_j. Then U_j is nonempty and Zariski open in V_i. Let $\Sigma_{\mathrm{TC}} V_i = V_i - \coprod U_j$. Then $\Sigma_{\mathrm{TC}} V_i$ is Zariski closed in V_i and has dimension $\leq d - i - 1$. Furthermore $V_i - \Sigma_{\mathrm{TC}} V_i$ is the disjoint union $\coprod U_j$ where each U_j is an analytic manifold of dimension $d - i$. Since $V_i \cap W_i^{d-i-1}$ is closed, $V_i - V_{i+1}$ is an open subset of $\coprod U_j$ and is thus also an analytic manifold of dimension $d - i$ without boundary.

One can check that the easy proof of [3, Theorem 1.7] goes through as before and consequently this new stratification is at least as good as the old one. However this new stratification has the additional property that the total codimension is constant on each component of $V_i - V_{i+1}$.

Now for the proof of the theorem. By transversality (and the codimension axiom, (TC3)) we may assume that the singularities of our k-parameter family of functions f_t have total codimension $\leq k$ for all t. Choose $s \geq 3$ so that all singularities of total codimension $\leq k + 1$ are s-determined and all "decomposable TB strata" of $J^s(n)$

(defined in [3]) have codimension $>k+1$. (It follows from the locally constant multiplicity axiom that the multiplicity of singularities of total codimension $\leqslant k+1$ is bounded, say by μ. By a well-known theorem of Mather [6] it follows that these singularities are all $(\mu+1)$-determined. Thus s exists.)

Let $\tilde{g}_t^u$ be a deformation of $j^s f_t|\partial D^k$ which lifts g_t^u (i.e., $g_t^u = j^{s(c)}\tilde{g}_t^u$). Let $\bar{g}_t^u$ be any generic extension of $\tilde{g}_t^u$ to all of D^k satisfying the following.

(1) $\bar{g}_t^1(M) \subseteq J_*^s(M)$ for all t,

(2) supp $\bar{g}_t^u \subseteq \operatorname{int} M$ for all t.

Since $\bar{g}_t^u$ is generic, it is a family of sections of $J_{k+1}^s(M)$. Let (J, p) be maximal in lexicographic order so that $p>c$ and the image of $\bar{g}_t^u$ has an element of total codimension p and TB index J. Then we will deform f_t and $\bar{g}_t^u$ so that the pair (J, p) is reduced in lexicographic order. After a finite number of steps f_t and $\bar{g}_t^u$ will be as desired.

Let $K \subseteq M \times D^k \times I$ be the set of all (x, t, u) so that $\bar{g}_t^u(x)$ has total codimension p and TB index J. Then by transversality K will be a compact smooth manifold with boundary $\partial K \subseteq M \times D^k \times 0$. Choose a Morse function $h: K \to I$ so that $h^{-1}(0) = \partial K$. Then we will reduce the number of critical points of h.

In order to remove a minimal critical point v of h of index $i+1$ we must realize a surgery on an i-sphere $S^i \subseteq \partial K$ by a deformation of f_t. The handle corresponding to v gives an $(i+1)$-disk along which S^i must be cancelled. By our dimensional hypotheses we have: $i+1 \leqslant k+1-p < k+1-c \leqslant n, k$, so $\pi_i O(k)$ is stable and the $(i+1)$-disk can be represented by a framed embedded disk $D^{i+1} \subseteq M^n \times D^k$ so that the projection $D^{i+1} \to D^k$ is a framed immersion (i.e., the first n vectors in the framing of D^{i+1} in $M \times D^k$ are tangent to M) (see [3, 9.11]). In a neighborhood of this $(i+1)$-disk we choose coordinates so that we can use our noninvariant refinement of Σ^J.

At this point we must deform f_t in a neighborhood of S^i and $\bar{g}_t^u$ in a neighborhood of D^{i+1} so that they are transverse to the noninvariant refinement of Σ^J near S^i and D^{i+1}. This will change the bad set K by an arbitrarily small isotopy in a neighborhood of D^{i+1}. Since the new K is diffeomorphic to the old one we will ignore the fact that it has changed. We also have a new S^i and D^{i+1} which are transverse to the stratification of Σ^J.

By [3, Theorem 2.2], we can deform f_t in a neighborhood of S^i so that $j^s f_t$ is fixed on the Σ^J points near S^i but f_t becomes equal to a polynomial function in $\Sigma^J + P = S^J$ where P is some fixed homogeneous polynomial of degree $s+1$. The deformation may create new $\Sigma^{J'}$ critical points where $J' < J$ but this is OK since we are doing downward induction on (J, P). Note that by the choice of s, the total codimension of $x \in \Sigma^J(f_t)$ is equal to the total codimension of $j_x^s(f_t) + P \in S^J$.

The jet map $\bar{g}_t^u | D^{i+1}$ gives a null homotopy of $\tilde{j}^s f_t | S^i$ in Σ^J. The jets in this null homotopy all have total codimension p. Thus we have a map $H: D^{i+1} \to S^J$. By an earlier deformation this is transverse to the stratification $\{V_i\}$ of S^J. As in [3] the desired elementary surgery on the bad set K can be accomplished by a sequence of surgeries on $\Sigma_i^J(f_t)$. $\square$

Now we want to state the C^1 local version of the higher singularities theorem. This will be the relative version of Theorem 2.1. It says that if the singularities of f_t already have total codimension $\leq c$ in some neighborhood of a compact set K and if the singularities of f_t in K have a certain additional structure which we call a "codimension-c homotopy framing", then there is an arbitrarily C^1 small deformation of f_t with support in the complement of K so that the singularities of total codimension $>c$ are eliminated.

The codim-c homotopy framings serve the same purpose as the framings of A_1 and A_2 points in the stable case (Section 1). They are defined as follows.

Definition 2.7. Let $f_t: M \to \mathbb{R}$, $t \in P^k$, be a smooth family of smooth functions which are nonsingular on ∂M. Then a *codimension-c homotopy framing* of f_t is a sequence of two-fold deformations $h^i_{t,u}$, $g^i_{t,u}$ ($t \in P$, $u \in I$, $i = 1, 2, 3, \ldots$) satisfying the following:

(1) $h^i_{t,u}$ is a deformation of f_t for all i;

(2) $g^i_{t,u}$ is a deformation of $j^s h^i_{t,1}$ for all i where $s \geq s(c)$;

(3) for any neighborhood U of $\Sigma(f)$, the deformations $h^i_{t,u}$ and $g^i_{t,u}$ have support in U for sufficiently large i;

(4) $h^i_{t,u}$ converges to f_t uniformly with respect to (t, u) in the C^1 topology as i goes to ∞;

(5) $h^i_{t,1}$ has only singularities of total codimension $\leq c$;

(6) $g^i_{t,u}(M) \subseteq J^s_c(M)$ for all i, u, t;

(7) $g^i_{t,1}(M) \subseteq J^s_*(M)$ for all i, t.

The notation comes from Theorem 2.6. For each i, $h^i_{t,u}$ and $g^i_{t,u}$ form a "two-fold deformation" of f_t since $f_t = h^i_{t,0}$ by (1) and $g^i_{t,0} = j^s h^i_{t,1}$ by (2). In the first stage the deformation $h^i_{t,u}$ takes f_t to a family of functions $h^i_{t,1}$ which, by (5), has only singularities of codimension $\leq c$. In the second stage the deformation $g^i_{t,u}$ is, by (7), a null homotopy of the s-jet of $h^i_{t,1}$ which, by (6), takes place inside the space $J^s_c(M)$ of all s-jets of codimension $\leq c$. Condition (4) says that for i large the deformation $h^i_{t,u}$ is an arbitrarily C^1 small perturbation of f_t. Condition (3) says that for i large the deformations $h^i_{t,u}$ and $g^i_{t,u}$ take place in an arbitrarily small neighborhood of the singular set of f_t.

If $h^i_{t,u}$ is the constant homotopy of f_t for all i, then we call $\{g^i_{t,u}\}$ a *strong homotopy framing* of f_t.

Proposition 2.8. *If $\{f_t, \xi_t\}$ is a family of framed functions, then f_t admits a strong codimension-1 homotopy framing.*

Proof. The space of "framed section" on M is contractible [5]. □

Proposition 2.9. (a) *$J^{s(c)}_c(n)$ is $(n+c-1)$-connected.*

(b) *The complement of $J^{s(c)}_c(M)$ in $J^{s(c)}(M)$ has codimension $>n+c$.*

(c) *If $k \leqslant c$, then a k-parameter family of function f_t on a compact n-manifold admits a codim-c homotopy framing.*

Proof. Since (a) implies (b) and (b) implies (c) it suffices to prove (a). But $J_c^{s(c)}(n)$ is weakly homotopy equivalent to the join $S^{n-1} * (J_c^{s(c)} \cap \mathcal{M}^2)$ where $\mathcal{M}^2$ is the square of the unique maximal ideal in $J^{s(c)}(n)$ (see [4]). Since $J_c^{s(c)}(n)$ is open in $J^{s(c)}(n)$, $J_c^{s(c)} \cap \mathcal{M}^2$ is locally path connected. Thus it suffices to show that $J_c^{s(c)} \cap \mathcal{M}^2$ is connected.

Let $\Sigma = \{p \in \mathcal{M}^2 \mid 0$ is an A_1 or A_2 singularity of $p\}$; then Σ is connected. In fact Σ has the n-homotopy type of BO (see [4]). But Σ is dense in $\mathcal{M}^2$ and thus also in $J_c^{s(c)}(n) \cap \mathcal{M}^2$ so $J_c^{s(c)} \cap \mathcal{M}^2$ is also connected. $\square$

Lemma 2.10. *Let K be a compact subspace of a Hausdorff space X. Let $\{U_j\}$ be a sequence of open neighborhoods of K in X so that $\bar{U}_j$ is compact, $U_j \supseteq \bar{U}_{j+1}$ and $\bigcap U_j = K$. Then given any neighborhood V of K, $U_j \subseteq V$ for sufficiently large j.*

Definition 2.11. Under the conditions of Lemma 2.10 we call the sequence $\{U_j\}$ an *infinitesimal neighborhood* of K in X.

Let $f_t : M \to \mathbb{R}$, $t \in P^k$, be a smooth family of smooth functions which are nonsingular on ∂M. Let $K \subseteq \Sigma(f_t)$ be a compact set. Let $\{h_{t,u}^i, g_{t,u}^i\}$ be a codim-$(k+1)$ homotopy framing of f_t. Then we say that $\{h_{t,u}^i, g_{t,u}^i\}$ has *codimension $\leqslant c$ outside of an infinitesimal neighborhood* of K if for every neighborhood V of K in $M \times D^k$ the set $\{(x, t, u) \in M \times D^k \times I \mid g_{t,u}^i(x) \notin J_c^s(M)\}$ is contained in V for sufficiently large i.

Theorem 2.12 (local higher singularities theorem). *Let $f_t : M \to \mathbb{R}$, $t \in D^k$, be a generic smooth family of smooth functions on a compact n-manifold and let K be a compact subset of $\Sigma(f_t)$. Suppose $k > n$ and $c = k - n + 1$. Let $\{h^i, g^i\} = \{h_{t,u}^i, g_{t,u}^i\}$ be a homotopy framing of f_t which has codimension $\leqslant c$ outside an infinitesimal neighborhood of K. Then there exists a codim-c homotopy framing $\{\bar{h}^j, \bar{g}^j\}$ of f_t and a strictly increasing sequence of positive integers $i(1) < i(2) < \cdots$ so that $\bar{h}^j$, $\bar{g}^j$ and $h^{i(j)}$, $g^{i(j)}$ are related as follows.*

> *For every neighborhood V of K in $M \times D^k$, $\bar{h}^j$, $\bar{g}^j$ are equal to $h^{i(j)}$, $g^{i(j)}$ outside V for sufficiently large j.* (*)

Proof. Since the proof will be by induction on k we need to make the statement of the theorem a little bit stronger so that the kth statement implies the $(k+1)$st statement.

We claim that if the homotopy framing $\{h^i, g^i\}$ has codimension c over a compact subset $L \subseteq D^k$, then the new homotopy framing $\{\bar{h}^j, \bar{g}^j\}$ can be chosen to agree with $\{h^{i(j)}, g^{i(j)}\}$ over an infinitesimal neighborhood of L. We should explain what we mean. We say that g^i has codimension c over a subset of D^k if $g_{t,u}^i(M) \subseteq J_c^s(M)$

whenever t lies in that subset. Since total codimension is upper semicontinuous this is an open condition on t. We say that $\{h^i, g^i\}$ has codimension c over $L \subseteq D^k$ if each g^i has codimension c over L. We say that $\{\bar{h}^j, \bar{g}^j\}$ and $\{h^{i(j)}, g^{i(j)}\}$ agree over an infinitesimal neighborhood $\{U_j\}$ of L in D^k if $\bar{h}^j_{t,u}$, $\bar{g}^j_{t,u}$ are equal to $h^i_{t,u}$, $g^i_{t,u}$ $(i = i(j))$ whenever $t \in U_j$.

We now proceed with the proof of this stronger statement. Let $\{U_j\}$ be an infinitesimal neighborhood of K in $(\text{int } M) \times D^k$. Then we will choose a large number $i(j)$ and deform $h^{i(j)}$, $g^{i(j)}$ with support in U_j to give $\bar{h}^j$, $\bar{g}^j$ as desired. The construction will be analogous to the construction in the proof of the local framed function theorem.

We have assumed that f_t is generic. Thus each singularity is finitely determined. For each $(x_0, t_0) \in K$ we choose an elevator (E_t, a, b, ϕ_t) parametrized over a neighborhood B of t_0 in D^k. As in the proof of the local framed function theorem we can construct a stack of elevators $E^1, \ldots, E^m$ over a neighborhood B of every point t_0 in D^k. These elevators will have the property that the E^i will be disjoint, $E^i \subseteq U_j$ for each i, and $\bigcup E^i \supseteq K \mid B$.

Choose a finite set of such sets B whose interiors cover D^k. Then choose a smooth triangulation of D^k so that each closed simplex lies in the interior of some B. Then we will prove the following statement by induction on p.

Induction hypothesis (p). There exists a homotopy framing $\{\bar{h}^e, \bar{g}^e\}$ of f_t and a strictly increasing sequence of positive integers $i(1) < i(2) < \cdots$ so that

(a) $\{\bar{h}^e, \bar{g}^e\}$ has codimension c over the p-skeleton of D^k,

(b) $\bar{h}^e$, $\bar{g}^e$ agree with $h^{i(e)}$, $g^{i(e)}$ outside U_j,

(c) $\{\bar{h}^e, \bar{g}^e\}$ and $\{h^{i(e)}, g^{i(e)}\}$ agree over an infinitesimal neighborhood of L.

This statement is trivial for $p = -1$ (just take $\bar{h}^e = h^e$, $\bar{g}^e = g^e$). Suppose that it is true for $p-1$. Let $\{\bar{h}^e, \bar{g}^e\}$ be as given by the induction hypothesis for $p-1$.

If $p < k$, then by induction on k the strengthened version of our theorem is true over each p-simplex of Δ^p. Consequently there is a codimension-c homotopy framing (H^d, G^d) of $f_t \mid \Delta^p$ and an increasing sequence $e(1) < e(2) < \cdots$ so that

(1) (H^d, G^d) and $(\bar{h}^{e(d)}, \bar{g}^{e(d)}) \mid \Delta^p$ agree outside an infinitesimal neighborhood of $K \mid \Delta^p$ in $(\text{int } M) \times \Delta^p$,

(2) (H^d, G^d) and $(\bar{h}^{e(d)}, \bar{g}^{e(d)}) \mid \Delta^p$ agree over an infinitesimal neighborhood of $(L \cap \Delta^p) \cup \partial \Delta^p$.

For each d consider the union $S_d \subseteq (\text{int } M) \times \Delta^p$ of the supports of $H^d - \bar{h}^{e(d)}$ and $G^d - \bar{g}^{e(d)}$. Then by (1), S_d lies in a small neighborhood of $K \mid \Delta^p$ and by (2), the projection $\text{pr}_2(S_d)$ of S_d in Δ^p is disjoint from $(L \cap \Delta^p) \cup \partial \Delta^p$. Let $\psi : \Delta^p \times D^{k-p} \to D^k$ be an embedding of a small normal disk bundle of Δ^p in D^k (take a normal half disk bundle $\Delta^p \times D^{k-p}_+$ if $\Delta^p \subseteq \partial D^k$). Then $\psi(\text{pr}_2(S_d) \times b)$ is disjoint from L and the $(p-1)$-skeleton of D^k for $\|b\|$ sufficiently small. This means we can paste in extensions of H^d, G^d into $\bar{h}^{e(d)}$, $\bar{g}^{e(d)}$ by the equation $\bar{H}^d_{t,u} = \bar{h}^e_{t,u} + \phi(b)(H^d_{a,u} - \bar{H}^e_{a,u})$ if $t = \psi(a, b)$ and $\bar{H}^d_{t,u} = \bar{h}^e_{t,u}$ if $t \notin \text{im } \psi$ where $e = e(d)$ and

$\phi : D^{k-p} \to I$ is a smooth function which is 1 at 0 and 0 in a neighborhood of ∂D^{k-p}. ($\bar{G}^d$ is defined similarly.) By choosing the normal disk bundle of Δ^p in D^k to be smaller and smaller as d increases we get a homotopy framing $\{\bar{H}^d, \bar{G}^d\}$ of f_t which has codimension c over L, Δ^p and the $(p-1)$-skeleton of D^k and which is equal to $\{h^{i(e(d))}, g^{i(e(d))}\}$ outside U_j. If we repeat this for each p-simplex, we eventually get a homotopy framing with codimension c over the entire p-skeleton as required by induction hypothesis (p).

Now take the case $p = k$. Let Δ^k be a k-simplex of D^k and let $\{\bar{h}^e, \bar{g}^e\}$ be a homotopy framing of f_t as given by the induction hypothesis for $p = k-1$. Let $E^1, \ldots, E^m$ be a stack of elevators lying over a base $B \supseteq \Delta^k$. Then by choosing e sufficiently large we may assume that $(\bar{h}^e, \bar{g}^e) | \Delta^p$ and $(h^{d(e)}, g^{d(e)}) | \Delta^k$ agree outside a compact subset of the interior of $\bigcup E^i | \Delta^k$ and that the analogous condition holds over every k-simplex of D^k. Now we apply Theorem 2.6 to $(\bar{h}^e, \bar{g}^e) | (E^i | \Delta^k)$. Then we get a new pair (H_i^e, G_i^e) which is equal to $(\bar{h}^e, \bar{g}^e)$ near $\partial(E^i | \Delta^k)$ and near $E^i | (\Delta^k \cap L)$. We define $(\bar{H}^e, \bar{G}^e)$ to be equal to (H_i^e, G_i^e) on $E^i | \Delta^k$ and $(\bar{H}^e, \bar{G}^e) = (\bar{h}^e, \bar{g}^e)$ outside $\bigcup E^i | \Delta^k$. Then $\bar{G}^e$ has codimension c over Δ^k. Repeating this over every k-simplex of Δ^k we eventually get $(\bar{H}^e, \bar{G}^e)$ with codimension c over all of D^k. Let $(\bar{h}^j, \bar{g}^j) = (\bar{H}^e, \bar{G}^e)$ and let $i(j) = e$. Then $\{\bar{h}^j, \bar{g}^j\}$ will be a homotopy framing of f_t satisfying all required conditions if $\bar{h}^j_{t,u}$ converges uniformly to f_t in C^1 norm of $\bar{h}^j_{t,u} - f_t$ bounded by $1/j$. For this we must "squeeze" the functions H_i^e in the elevators $E^i | \Delta^k$ over the complement of $(\Delta^k \cap L) \cup \partial \Delta^k$ as in the local framed function theorem. Since the squeezing construction changes singularities by left scalar multiplication we must assume that total codimension is scalar invariant. □

We give three examples of how the local higher singularities theorem can be used. The proofs are similar to the proofs of the corresponding statements in Section 1. We give the proof only in the last case.

Corollary 2.13. *Let $M^n \to E^{n+k} \to B^k$ be a smooth bundle of paracompact smooth manifolds of the indicated dimensions where $k > n$. Then there is a smooth function $f : E \to \mathbb{R}$ so that on each fiber M, $f | M$ is nonsingular on ∂M and has only singularities of total codimension $\leq k - n + 1$.*

Proof. Analogous to the proof of Corollary 1.7. □

Corollary 2.14. *Let M^n be a compact smooth n-manifold. Let $f_t : M \to S^1$, $t \in D^k$, be a smooth family of smooth functions which are nonsingular on ∂M and suppose $k > n$. Suppose that f_t has the structure of a smooth family of framed circle functions for t near ∂D^k. Then there is a smooth family of functions $f_t' : M \to S^1$ so that f_t' is arbitrarily C^1 close to f_t, $f_t' = f_t$ near ∂D^k and f_t' has only singularities of total codimension $\leq k - n + 1$.*

Proof. Analogous to the proof of Corollary 1.8. □

Corollary 2.15. *Let M be a compact smooth n-manifold and let $\mathcal{F}$ be a smooth foliation of M with codimension $q > \frac{1}{2}n$. Let $f: M \to \mathbb{R}$ be a smooth function satisfying the following boundary condition:*

> *If $x \in \partial M$ and the derivative of f at x is trivial on the tangent plane to the leaf at x, then $\mathcal{F}$ is tangent to ∂M in a neighborhood of x.*

Then f can be approximated by a smooth function f' which satisfies this same boundary condition so that f' has only singularities of total codimension $\leq 2q - n + 1$ on each leaf of $\mathcal{F}$. Furthermore, f' can be chosen so that the first derivative of $f' - f$ is uniformly small in the direction tangent to the leaves of $\mathcal{F}$.

Remark. (a) The analogous result holds for maps $M \to S^1$.

(b) If $\mathcal{F}$ is tangent to ∂M, then any function $f: M \to S^1$ satisfies the boundary condition so f' always exists.

Proof. (This is analogous to the proof of Corollary 1.10.) First make f generic. Then $\Sigma(f)$ is a compact smooth q-manifold. Cover $\Sigma(f)$ with a finite number of product neighborhoods $N_1, \ldots, N_m$. By the local higher singularities theorem there is a homotopy framing $\{h^i, g^i\}$ of $f_t \mid N_1$ of total codimension c. Now go to N_2. Passing to a subsequence of $\{h^i, g^i\}$ if necessary, we have a partial homotopy framing of $f_t \mid N_2$ of total codimension c. By the local higher singularities theorem we can extend this to all of N_2. Continuing in this way we get a compatible system of homotopy framing of $f_t \mid N_j$ for all j. Then $f_t' = h^i_{t,1}$ is the desired function. □

3. The C^0 approximation theorem

Let $p: M \to \mathbb{R}$ be a fixed function on a compact smooth manifold M so that p is nonsingular on ∂M. Then Chaltin showed that any k-parameter family of functions $f_t: M \to \mathbb{R}$, $t \in D^k$, can be C^0 approximated by a k-parameter family of generalized Morse functions. In fact Chaltin's idea proves a slightly stronger statement.

Theorem 3.1. *Let M, P be compact smooth manifolds and let $f_t: M \to \mathbb{R}$, $t \in P$, be a smooth family of smooth functions which are equal to the fixed function p near ∂M. (In particular f_t is nonsingular on ∂M for all t in P.) Then f_t can be C^0 approximated by a smooth family of Morse functions f_t' which are equal to p near ∂M.*

Remark. A relative version of this theorem is needed to make useful applications analogous to the corollaries in Sections 1 and 2.

Proof. To approximate f_t by a family of Morse functions f_t' we will add a fixed function h. (Thus $f_t' = f_t + h$ for all t.) If h is very "bumpy", then f_t' will have only nondegenerate singularities as illustrated by the graphs stated in Fig. 1. Since M and P are compact the first and second derivatives of f_t are bounded. Thus if either the first or second derivative of h is sufficiently large at each point, then $f_t' = f_t + h$ will have only nondegenerate singularities.

More precisely the requirements for the function h are as follows. Let U be an open neighborhood of ∂M so that $f_t | U = p | U$ for all t. (We can perturb f_t so that such a U exists.) Let B and C be the maximal values of $\|Df_t(x)\|$ and $\|D^2 f_t(x)\|$ for all $(x, t) \in M \times P$ (with respect to some fixed metric on M) and let $\varepsilon > 0$. Then h needs to satisfy the following conditions:

(1) $h + p$ is a Morse function,
(2) $h = 0$ near ∂M,
(3) $|h(x)| < \varepsilon$ for all $x \in M$,
(4) for each $x \in M - U$ either
  (a) $\|Dh(x)\| > B$ or
  (b) $\|D^2 h(x)(u)\| > C\|u\|$ for all nonzero vectors $u \in T_x M$ = the tangent plane to M at x.

If condition (4)(a) holds, the $D(f_t + h)(x) \neq 0$ for all t and condition (4)(b) would imply that $D^2(f_t + h)(x)$ is nonsingular for all t. Thus $f_t + h$ will have only nondegenerate singularities on $M - U$ for all t. Condition (1) implies that $f_t + h$ will have only nondegenerate singularities on U for all t. Condition (2) implies that $f_t + h = p$ near ∂M and condition (3) implies that $f_t + h$ is ε-close to f_t for all t.

Before we construct h we note that conditions (1) and (2) are unnecessary. In other words we claim that if there is a function h satisfying (3) and (4), then there is another function h' satisfying (1)–(4). To see this we first construct $h'': M \to \mathbb{R}$ satisfying (2)–(4) by multiplying the function h by any smooth function $\phi: M \to [0, 1]$ so that $\phi = 0$ near ∂M and $\phi = 1$ on $M - U$. Then we perturb $h'' + p$ with support in int M so that it becomes a Morse function. Subtracting p we get h' as desired. Thus it suffices to construct a smooth function $h: M \to (-\varepsilon, \varepsilon)$ satisfying condition (4).

If $n = 1$, then a modified sine function will do the trick so suppose that $n > 1$ and the statement (the existence of h) holds for any manifold of dimension $< n$. Now choose any Morse function $g: M \to [0, 1]$ so that $g^{-1}(0) = \partial M$. Let $f: \mathbb{R} \to [-\frac{1}{4}\varepsilon, \frac{1}{4}\varepsilon]$ be a modified sine function $f(y) = \frac{1}{4}\varepsilon \sin(ay + b)$ having the property that $\|f(c)\| < \frac{1}{8}\varepsilon$

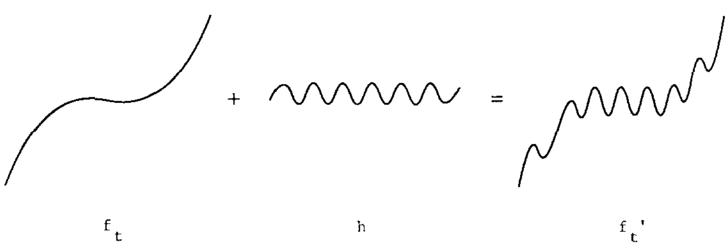

Fig. 1. $f_t + h = f_t'$.

for every critical value c of g and so that the composition fg satisfies condition (4) on $(fg)^{-1}[-\frac{1}{8}\varepsilon, \frac{1}{8}\varepsilon]$.

For each critical point y of f let $L_y = g^{-1}(y)$. Then L_y is a closed $(n-1)$-manifold and there is a tubular neighborhood of L_y diffeomorphic to $L_y \times (-\delta, \delta)$ for some small $\delta > 0$ so that g is given by $g(x, s) = y + s$ for $(x, s) \in L_y \times (-\delta, \delta)$. By induction on n there is a smooth function $h_y : L_y \to (\frac{1}{2}\varepsilon, \varepsilon)$ so that at each point in L_y one of the first two derivatives of h_y is very large. Depending on the sign of $f(y)$, we can extend $\pm h_y$ to a function $\bar{h}_y$ on a small neighborhood of $L_y = L_y \times 0$ in $L_y \times (-\delta, \delta)$ by $\bar{h}_y(x, s) = \pm(h_y(x) - s^2)$. Since the value of $\bar{h}_y$ is much larger in absolute value than the value of fg we can paste together the functions $\bar{h}_y$ and fg to get a new function h which is equal to fg on $(fg)^{-1}[-\frac{1}{8}\varepsilon, \frac{1}{8}\varepsilon]$ and equal to $\bar{h}_y$ in a small neighborhood of L_y and so that $\|Dh\| > B$ in the rest of M. This is the desired function. $\square$

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